2.6.50 A light situated at a point in a plane sends out beams of light in all directions. The beams in the plane meet a curve and are all reflected parallel to a line in the plane, as shown in Figure 9.

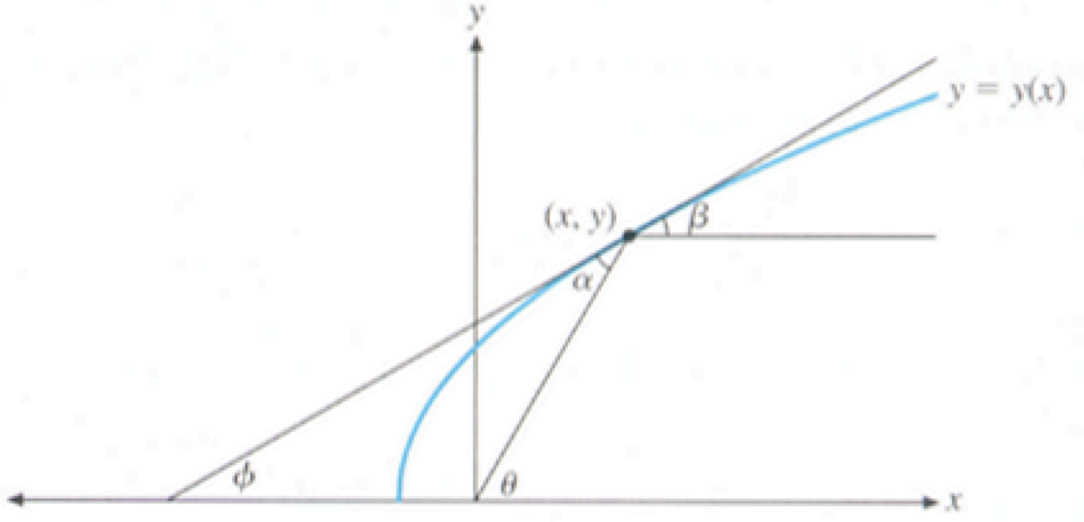


Figure 9. The reflector in Exercise 50.

The light is reflected so that the angle of incidence α equals the angle of reflection β.

a) Show that tan(ϑ) = tan(2β); then use trigonometry to show that

$$\frac{y}{x} = \frac{2y'}{1-(y')^2}$$

ERROR IN THE TEXT. THE EQUATION ABOVE IS CORRECT!

from Fig. 9: α = β (given), φ = β (corresponding angles for parallel lines), and ϑ = 2β (external angle of a triangle is equal to the sum of the two non-adjacent interior angles)

$$so,\ tan(\theta) = tan(2\beta) = \frac{2\ tan(\beta)}{1-tan^2(\beta)}$$

$$y' = tan(\phi) = tan(\beta)$$

$$tan(\theta) = \frac{2\ tan(\beta)}{1-tan^2(\beta)} = \frac{2y'}{1-(y')^2} = \frac{y}{x}$$

b) Use the quadratic formula to solve the equation for y'; then solve the resulting first-order ODE to find the equation of the reflecting curve.

$$2xy' = y - y(y')^2$$

$$y(y')^2 + 2xy' - y = 0$$

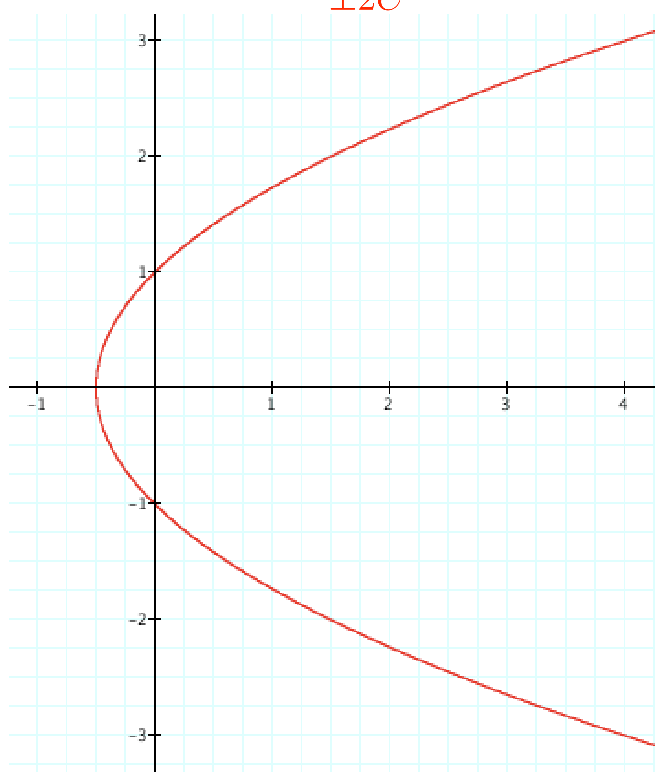
$$y' = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

$$\left(-x \pm \sqrt{x^2+y^2}\right) dx - y\ dy = 0$$

This can be solved as an exact ODE with the integrating factor: $\frac{1}{\sqrt{x^2+y^2}}$

$$\left(\frac{x}{\sqrt{x^2+y^2}} \pm 1\right) dx + \left(\frac{y}{\sqrt{x^2+y^2}}\right) dy = 0$$
$$F(x,y) = \sqrt{x^2+y^2} \pm x + \phi(y)$$
$$\phi'(y) = 0 \rightarrow \phi = k$$
$$F(x,y) = \sqrt{x^2+y^2} \pm x = C$$
$$x^2 + y^2 = (C \pm x)^2 = C^2 \pm 2Cx + x^2$$
$$y^2 = C^2 \pm 2Cx$$
$$y = \pm\sqrt{C^2 \pm 2Cx}$$

$$x = \frac{y^2 - C^2}{\pm 2C}$$



$$\left(-x \pm \sqrt{x^2+y^2}\right) dx - y\, dy = 0$$

This can also be solved as a homogeneous equation of degree 1.

$$y = ux \qquad dy = udx + xdu$$

$$(x \pm \sqrt{x^2 + u^2x^2})dx + ux(udx + xdu) = 0$$

$$(x \pm \sqrt{x^2 + u^2x^2} + u^2x)dx = -ux^2du$$

$$\frac{1}{x}dx = -\frac{u}{1 \pm \sqrt{1+u^2} + u^2}du$$

Let $w = \sqrt{1+u^2}$, then $dw = \frac{u}{\sqrt{1+u^2}}du$

$$\frac{1}{x}dx = -\frac{u}{1 \pm \sqrt{1+u^2} + u^2}du = -\frac{1}{\pm 1 + w}dw$$

$$ln|x| = -ln|\pm 1 + w| + C$$

$$x = \frac{A}{\pm 1 + w} = \frac{A}{\pm 1 + \sqrt{1+u^2}} = \frac{A}{\pm 1 + \sqrt{1 + \frac{y^2}{x^2}}}$$

$$\pm x + \sqrt{x^2 + y^2} = A$$

$$x^2 + y^2 = (A \pm x)^2 = A^2 \pm 2Ax + x^2$$

$$y = \pm\sqrt{A^2 \pm 2Ax}$$

$$x = \frac{y^2 - A^2}{\pm 2A}$$

Same solution as before.