Problem 5.6.9

5.6.9 Consider the equation

$$x'' + 2x' + 2x = \delta(t), x(0) = x'(0) = 0$$

a) Use the fact that $L\{\delta(t)\} = 1$ to show that the solution is $x(t) = e^{-t}sin(t)$.

$$s^2X + 2sX + 2X = 1$$

$$X(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1} \rightarrow x(t) = e^{-t}sin(t)$$

b) Show that the solution of

$$x'' + 2x' + 2x = \delta(t,\epsilon), x(0) = x'(0) = 0$$

is

$$x_\epsilon(t) = \frac{1}{2\epsilon}\begin{cases} 1 - e^{-t}\left(cos\ t + sin\ t\right) & if\ 0 \le t < \epsilon \\ -e^{-t}\left(cos\ t + sin\ t\right) + e^{-(t-\epsilon)}\left(cos(t-\epsilon) + sin(t-\epsilon)\right) & if\ \epsilon \le t \end{cases}$$

$$s^2X + 2sX + 2X = \left(\frac{1-e^{-s\epsilon}}{s\epsilon}\right)$$

$$X(s) = \frac{\left(\frac{1-e^{-s\epsilon}}{s\epsilon}\right)}{s^2+2s+2} = \left(\frac{1-e^{-s\epsilon}}{s\epsilon}\right)\frac{1}{(s+1)^2+1}$$

$$\frac{1}{(s+1)^2+1} = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+1}$$

$$1 = A(s^2+2s+2) + Bs^2 + Cs$$

$$A = \frac{1}{2} \qquad B = \frac{-1}{2} \qquad C = -1$$

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$$\frac{1}{(s+1)^2+1} = \frac{1}{2}\left(\frac{1}{s} - \frac{s+2}{(s+1)^2+1}\right) = \frac{1}{2}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\right)$$

$$X(s) = \frac{1}{2\epsilon}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\right) - \frac{e^{-s\epsilon}}{2\epsilon}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\right)$$

$$x(t) = \frac{1}{2\epsilon}\left\{\left[1 - e^{-t}\left(cos\ t + sin\ t\right)\right] H(t) - \left[1 - e^{-(t-\epsilon)}\left(cos\ (t-\epsilon) + sin\ (t-\epsilon)\right)\right] H(t-\epsilon)\right\}$$

$$x_\epsilon(t) = \frac{1}{2\epsilon}\begin{cases} 1 - e^{-t}\left(cos\ t + sin\ t\right) & if\ 0 \le t < \epsilon \\ -e^{-t}\left(cos\ t + sin\ t\right) + e^{-(t-\epsilon)}\left(cos(t-\epsilon) + sin(t-\epsilon)\right) & if\ \epsilon \le t \end{cases}$$

c) Use l'Hospital's rule to argue that the solution of part(b) approaches that of part(a) as $\epsilon \to 0$, at least for $t > 0$. (Let $t = k\epsilon$, where $0 \le k < 1$ for the first part.) Note that $e^t \to 1 + t, cos\ t \to 1$ and $sin\ t \to t$ as $t \to 0$.

$$\lim_{\epsilon\to 0} \frac{1 - e^{-k\epsilon}\left(cos(k\epsilon) + sin(k\epsilon)\right)}{2\epsilon} \to \lim_{\epsilon\to 0} \frac{1 - (1-k\epsilon)(1+k\epsilon)}{2\epsilon} \to \lim_{\epsilon\to 0} \frac{k^2\epsilon}{2} \to 0$$

$$\lim_{\epsilon\to 0} \frac{-e^{-t}(cos\ t + sin\ t) + e^{\epsilon}e^{-t}\left[cos(t-\epsilon) + sin(t-\epsilon)\right]}{2\epsilon} \to \frac{0}{0}$$

$$\lim_{\epsilon\to 0} \frac{e^{\epsilon}e^{-t}\left[cos(t-\epsilon) + sin(t-\epsilon)\right] + e^{\epsilon}e^{-t}\left[sin(t-\epsilon) - cos(t-\epsilon)\right]}{2} \to e^{-t}sin\ t$$